Distinguishability dynamics and quantum speed limits in the Bloch ball

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We derive a necessary and sufficient condition on a time-independent Hamiltonian and arbitrary initial state of a two-level quantum system which guarantees that the initial state will time evolve to another state from which it can be distinguished with an *a priori* established minimal error probability $\delta \in [0, 1/2]$. The time required for the initial state to evolve to such a probabilistically distinguishable state is compared to previously established quantum speed limits giving lower bounds on the time required for evolution between nonorthogonal states. We use the explicit distinguishability time to provide a succinct derivation of the quantum brachistochrone, i.e., the Hamiltonian generating shortest time evolution between two states having equal magnitude Bloch vectors. As a consequence of the distinguishability condition, we find that if a given unitary does not evolve a given pure state to an orthogonal state, then there exist mixed states in the Bloch ball that time evolve faster than the given pure state and that reach a greater level of distinguishability under the given unitary. We show that such mixed states exist even in the case of nonunitary evolution by considering a two-level system resonantly coupled to a single mode electromagnetic field.

I. INTRODUCTION

The minimal length of time required for a given quantum state to evolve to an orthogonal state under unitary time evolution provides an ultimate bound for the processing speed of a quantum computer, regardless of the physical substrate used for the quantum information processing [1]. Orthogonal states also form a valuable resource for quantum communication and for the efficiency of quantum algorithms [2]. However, in practice, perfectly orthogonal states are not always achievable; for this reason, it is not surprising that the problem of optimally distinguishing elements of a set of nonorthogonal quantum states continues to be subject of active research (see, e.g., Refs.[3, 4]). Methods for generation and manipulation of nonorthogonal states are vital for high precision control of quantum dynamics and for optimal covariant quantum state estimation [5]. The unavoidability of nonorthogonal initial and final states of realistic quantum dynamics has led to the study of generalized quantum speed limits, i.e., lower bounds on the minimal time required for an initial state to evolve into a state which is imperfectly distinguishable from the initial state [6, 7].

In this paper we consider the more general question of determining a simple necessary and sufficient condition on a given time-independent Hamiltonian H and initial qubit state ρ indicating that the unitary time evolution generated by H produces a state that is probabilistically distinguishable (with maximal success probability $1-\delta$) from ρ. The proof of such a condition requires a notion of distinguishability time, which we briefly define. In order to operationally define a distinguishability time $\tau_\delta(\rho, \mathcal{E}_t, \Delta)$, representing the minimal time required for an initial quantum state ρ to time evolve to a state from which it is probabilistically distinguishable with error δ, one must specify in addition to a parametrized quantum dynamical map $\mathcal{E}_t$, a discrimination prodecure Δ that maps the pair $(\rho, \mathcal{E}_t(\rho))$, consisting of initial state ρ and its time evolved counterpart $\mathcal{E}_t(\rho)$, to the interval $[0, 1/2]$. This interval represents the probability of unsuccessful discrimination events. Then one can define

$$\tau_\delta(\rho, \mathcal{E}_t, \Delta) := \min\{t | \Delta(\mathcal{E}_t\rho, \rho) = \delta\}. \quad (1)$$

Importantly, in order for $\tau_\delta(\rho, \mathcal{E}_t, \Delta)$ to be well-defined, there must exist a time $t > 0$ such that $\Delta(\rho, \mathcal{E}_t\rho) = \delta$. The calculation of $\tau_\delta(\rho, \mathcal{E}_t, \Delta)$ for a given time-evolution and discrimination procedure may require solving for the full path $\{\mathcal{E}_t\rho | t \geq t_0\}$ and optimizing over a set of quantum measurements determined by the decision procedure Δ. The general solution of this poses a considerable challenge. However, determination of a range of precisions δ in $[0, 1/2]$ such that $\tau_\delta(\rho, \mathcal{E}_t, \Delta)$ is meaningful can be obtained from, e.g., an upper bound on $\Delta(\rho, \mathcal{E}_t\rho)$ for all t. In addition, it is useful to derive tight lower bounds on τ_δ that reveal the physical properties of the state and the dynamics that lead to the true value of the distinguishability time.

The present definition of τ_δ is easily generalizable to the case of optimal multistate distinguishability dynamics [4] and to maps $\mathcal{E}_t$ parametrized by an arbitrary smooth manifold instead of a line. The discrimination procedure Δ implicitly depends on a set of available measurements and postprocessing of measurement results. We define a symmetric relation on the set of quantum states termed "$(1-\delta)$-distinguishability" which will be useful in the discussion that follows by the following relationship: ρ and σ are $(1-\delta)$-distinguishable if and only if $\Delta(\rho, \sigma) = \delta$.

In the present work, we focus primarily on the case in which $\mathcal{E}_t$ is unitary evolution generated by a time-independent Hamiltonian, $\{\mathcal{E}_t(\rho) | t \geq 0\}$ is a path in the quantum state space of a two-level system, and Δ is the minimal error probability for binary quantum state discrimination [8]. Not all Hamiltonians generate a unitary time-evolution taking a given initial state to an orthog-

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onal state or even to a $(1-\delta)$-distinguishable state for a given value of δ (results along these lines for various finite dimensional Hilbert spaces can be found in Ref. [9]). Our main theorem completely determines the relationship between distinguishability of states on a unitary path in the Bloch ball and the statistical geometry of that path. Given a Hamiltonian and a distinguishability parameter $\delta \in [0, 1/2]$, the theorem allows qubit states to be split into two classes: 1) those initial states that, under unitary evolution generated by H, become $(1-\delta)$-distinguishable from the starting point, and 2) those initial states that do not. For the first class, the theorem allows one to introduce a partial order on the states based on the minimal time required to get to $(1-\delta)$-distinguishability, i.e., the quantum speed limits of their respective orbits in the Bloch ball. As a consequence, we determine the subset of the Bloch ball that reaches $(1-\delta)$-distinguishable states faster than a given pure state under the evolution generated by an arbitrary time-independent qubit Hamiltonian.

A brief outline of the paper is as follows: in Section II, we review two lower bounds on the distinguishability time when Δ is taken to be the minimum error binary quantum state discrimination procedure and compare these bounds to the true unitary distinguishability time of a qubit. Section III contains the main theorem providing a necessary and sufficient condition for evolution to a $(1-\delta)$-distinguishable state and an explicit expression for the distinguishability time of a two-level system. We use the main theorem to provide a short solution to the problem of the quantum brachistochrone in Section IV. In Section V, we extend the analysis to nonunitary qubit dynamics and show that a set of mixed states evolving to $(1-\delta)$-distinguishable states faster than a given pure state exists even in the presence of amplitude damping due to coupling to a monochromatic electromagnetic field. We also use this example to illustrate the role of the initial field state in controlling the minimal distinguishability time, showing how this can be tailored to speed up arrival to a target level of distinguishability.

II. DISTINGUISHABILITY TIME

Rigorous notions of uncertainty tradeoffs between measurements of energy and time have been developed in terms of the orthogonalization time, i.e., the minimum time required for an initial quantum state to evolve under the action of a given (unitary or nonunitary) quantum dynamical map to a state from which it is completely distinguishable [7, 10–12]. In traditional approaches to time-energy uncertainty, a decay time or half-life of a quantum system scales inversely with the root mean square energy fluctuations of the system. These approaches were made mathematically rigorous by the derivation of a lower bound on the pure state orthogonalization time which scales inversely with the variance of the generator of evolution (we call this bound $\mathrm{MT}_\perp$ after the seminal work of Mandelstam and Tamm [13] which was put on a geometric footing by Aharanov and Anandan [14]).

The orthogonalization time can also be bounded below by a function of the expected value of the generator of evolution (we call such a bound $\mathrm{ML}_\perp$ after Margolus and Levitin [15]). An important difference between the $\mathrm{ML}_\perp$ and $\mathrm{MT}_\perp$ bounds is that the former is a kinematic bound, resulting from a linear approximation to the fidelity of the initial state and the time-evolved state, while the latter was derived by consideration of geodesics of an appropriate metric on quantum state space.

Recently, the $\mathrm{ML}_\perp$ and $\mathrm{MT}_\perp$ bounds have been generalized to bounds on the distinguishability time of Eq.(1) for general quantum states evolving under unitary maps [6]. When the unitary path is generated by $H = H^\dagger$ and $\Delta(\rho, \sigma) := p_{\mathrm{err}}(\rho, \sigma) = 1/2 - 1/4\|\rho - \sigma\|_1$, where $p_{\mathrm{err}}(\rho, \sigma)$ is the minimal error probability for binary state discrimination of quantum states ρ and σ [8], the time t required for ρ to reach a $(1-\delta)$-distinguishable state is bounded below by the following distinguishability times:

$$\tau_\delta^{MT} = \frac{2\sin^{-1}(1-2\delta)}{\sqrt{\mathcal{F}(\rho, H)}} \tag{2}$$

$$\tau_\delta^{ML} = \frac{\pi\hbar(1-\sqrt{1-(1-2\delta)^2})}{2(\mathrm{tr}(\rho H) - E_0)} \tag{3}$$

where E_0 is the least eigenvalue of H and $\mathcal{F}(\rho, H)$ is the quantum Fisher information on the unitary path containing ρ and generated by H (see Ref.[8] or Section III for a definition). Clearly, the unified distinguishability bound satisfies $\lim_{\delta\to 0}\max\{\tau_\delta^{MT}, \tau_\delta^{ML}\} = \max\{\mathrm{MT}_\perp, \mathrm{ML}_\perp\}$. For a two-level system with time-evolution generated by $H = \hbar\omega_0 \vec{n}\cdot\vec{\sigma}$, we will see in Section III that the states $c_1|0\rangle_{\vec{n}} + c_2|1\rangle_{\vec{n}}$ with $|c_1| = |c_2| = 1/\sqrt{2}$ are the only ones that saturate τ_δ^{MT}; not surprisingly, these are also the only states saturating the $\mathrm{ML}_\perp$ and $\mathrm{MT}_\perp$ bounds [15, 16].

For a two-level system in a state $\rho(\vec{r}) = \mathbb{I} + \vec{r}\cdot\vec{\sigma}/2$ evolving by the Hamiltonian $H = \hbar\omega_0(\vec{n}\cdot\vec{\sigma} + \mathbb{I})$ (the identity is added so that H has positive semidefinite spectrum), application of Eq.(3) yields

$$\tau_\delta^{ML} = \frac{\pi(1-\sqrt{1-(1-2\delta)^2})}{2\omega_0(\vec{n}\cdot\vec{r}+1)}. \tag{4}$$

This bound is not consistent with the rotational symmetry of the dynamics. For example, the state with Bloch vector $\rho(-\vec{r})$ gives a different bound than that for ρ. The bound τ_δ^{ML} remains valid if $\vec{n}\cdot\vec{r}$ is replaced by $|\vec{n}\cdot\vec{r}|$. However, our explicit calculation of $\mathcal{F}(\rho, H)$ in Section III, combined with the fact that $\sin^{-1}(1-2\delta) \geq (\pi/2)(1-\sqrt{1-(1-2\delta)^2})$ for $\delta \in [0, 1/2]$, leads to the conclusion that for $\delta \in (0, 1/2)$, the Mandelstam-Tamm bound τ_δ^{MT} is greater than the Margolus-Levitin bound τ_δ^{ML}. Hence we will focus here on τ_δ^{MT} as the lower bound on the distinguishability time for the two-level

system. Similar to $\mathrm{ML}_\perp$, we expect τ_δ^{ML} to be useful for analysis of the distinguishability dynamics of systems with constant energy but large quantum Fisher information, which for pure states is equivalent to large fluctuations of the energy. One potentially important application of the quantity τ_δ^{ML} is thus for analysis of distinguishability dynamics in incompressible liquids with large heat capacity near thermal phase transitions.

III. SINGLE QUBIT DISTINGUISHABILITY DYNAMICS

Before we state the main theorem, we make note of some of its corollaries for pure states which are well-known. Consider a Hamiltonian $H = \hbar\omega_0 \vec{n}\cdot\vec{\sigma}$ (where $\|\vec{n}\| = 1$ and $\vec{\sigma} := (\sigma_x, \sigma_y, \sigma_z)$) and an initial state $|\psi\rangle$. H has operator norm $\hbar\omega_0$ and we will set $\omega_0 = 1$ for convenience. By acting on $|\psi\rangle$ with the time-evolution operator $U(t) := e^{-iHt/\hbar}$ to produce $|\psi(t)\rangle$, one finds that a time t such that $\langle\psi|\psi(t)\rangle = 0$ exists if and only if the Bloch vector $\vec{r}$ representing $|\psi\rangle$ on the Bloch sphere is orthogonal to $\vec{n}$. These states are superpositions $|\phi(\varphi)\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{\vec{n}} + e^{i\varphi}|1\rangle_{\vec{n}})$ of the lowest and highest energy states ($|1\rangle_{\vec{n}}$ and $|0\rangle_{\vec{n}}$, respectively) of H. Such superpositions define a great circle of states on the Bloch sphere having Bloch vector orthogonal to $\vec{n}$. A measurement of the observable H in a state $|\phi(\varphi)\rangle$ has variance 1, the largest possible value for all pure states. For any φ, the state $U(t)|\phi(\varphi)\rangle$ is orthogonal to $|\phi(\varphi)\rangle$ when $t = \pi/2$.

On the other hand, there are no completely distinguishable mixed states (i.e., nontrivial convex combinations of pure states) in the Bloch ball. Furthermore, if the initial state is mixed, it cannot be distinguished completely from any pure state. Mathematically, these facts follow from the fact that mixed states of the Bloch ball have rank two and so they cannot have support which is disjoint from the support of any other state of the Bloch ball. Hence, when the initial state is mixed, there is no hope to achieve 1-distinguishability through any type of evolution, unitary or nonunitary. However, the evolution may still result in a $(1-\delta)$-distinguishability of initial and final states for some $\delta > 0$. Here, we consider unitary evolutions which result in $(1-\delta)$-distinguishability between the initial and final states instead of perfect distinguishability (i.e., 1-distinguishability) and derive the set of quantum states which evolve to $(1-\delta)$-distinguishability faster than a given pure state. In the finite dimensional case considered here, the condition of $(1-\delta)$-distinguishability of two qubit states is made easier by the fact that the trace norm $\|\cdot\|_1$ appearing in the expression $p_{\mathrm{err}}(\rho,\sigma)$ can be calculated as the sum of the absolute values of the eigenvalues of $\rho - \sigma$. For the statement of the main theorem, we again take $H = \hbar\omega_0\vec{n}\cdot\vec{\sigma}$ and $\omega_0 = 1$. The proof is made easier by the use of a simple lemma.

Lemma Let $\rho_0 = \frac{\mathbb{I}}{2} + \frac{\vec{r}_0\cdot\vec{\sigma}}{2}$ be an initial quantum state on the unitary path $\rho_t = e^{-iHt/\hbar}\rho_0 e^{iHt/\hbar}$ generated by H. Then the quantum Fisher information on this path satisfies

$$\mathcal{F}(\rho_t, H) = 4\|\vec{n}\times\vec{r}_0\|^2. \tag{5}$$

for all $t \geq 0$.

Proof By definition, $\mathcal{F}(\rho_t, H) := \mathrm{tr}(L^2\rho_t)$, where $L = L^\dagger$ is the symmetric logarithmic derivative operator, i.e., the unique observable satisfying $\frac{d\rho_t}{dt} = \frac{1}{2}(L\rho_t + \rho_t L)$ for all t. L depends on the state ρ_t through its Bloch vector $\vec{r}_t$ and also on the vector $\vec{n}$ defining H. The von Neumann equation gives $\frac{d\rho_t}{dt} = -i/\hbar[H,\rho_t]$, so that L must satisfy $\frac{1}{2}[L,\rho_t]_+ = -i/\hbar[H,\rho_t] = (\vec{n}\times\vec{r}_t)\cdot\vec{\sigma}$ for all t. Writing $L = \vec{v}_t\cdot\sigma$ and solving for $\vec{v}_t$ results in $\vec{v}_t = 2(\vec{n}\times\vec{r}_t)$. Taking the variance of the resulting operator $L = 2(\vec{n}\times\vec{r}_t)\cdot\vec{\sigma}$ in the state ρ_t and using that fact that $\mathrm{tr}(\rho L) = 0$ gives the result $\mathcal{F}(\rho_t, H) = 4\|\vec{n}\times\vec{r}_t\|^2$. The Bloch vector $\vec{r}_t$ satisfies a quantum equation of motion that is the same as the classical equation of motion for a magnetic moment in a constant magnetic field which gives rise to Larmor precession, and so $\|\vec{n}\times\vec{r}_t\| = \|\vec{n}\times\vec{r}_0\|$ for all t. In this classical analogy, the constant value of the norm corresponds to conservation of angular momentum. □

Because the observable L has units of $[t]^{-1}$, $\mathcal{F}(\rho, H)$ has units of $[t]^{-2}$. The geometric relationships among the symmetric logarithmic derivative, the Hamiltonian, and the state ρ can be seen in Figure 1a). More general expressions for the symmetric logarithmic derivative and quantum Fisher information for a qubit have been obtained previously [17], but for the case of unitary evolution the vectorial expression is exceptionally useful. In the context of time-dependent quantum magnetometry with a collection of qubits, Eq.(5) reproduces the relevant quantum Fisher information appearing in the quantum Cramér-Rao bound [18]. Clearly, when $\|\vec{r}\| = 1$, i.e., when ρ is pure, the quantum Fisher information takes the well known value $\mathcal{F}(\rho, H) = 4\mathrm{tr}((\Delta H)^2)\rho)/\hbar^2$ [19]. We now state and prove the main theorem.

Theorem 1 There exists a $t \geq 0$ such that an initial quantum state $\rho = \frac{1}{2}(\mathbb{I} + \vec{r}\cdot\vec{\sigma})$ evolves to a state $U(t)\rho U(t)^\dagger$ satisfying $p_{\mathrm{err}}(\rho, U(t)\rho U(t)^\dagger) = \delta$ if and only if:

$$2(1-2\delta) \leq \sqrt{\mathcal{F}(\rho, H)}. \tag{6}$$

where $\|\vec{a}\| = (\vec{a}\cdot\vec{a})^{1/2}$ is the Euclidean norm of $\vec{a}\in\mathbb{R}^3$.

Proof It follows from the algebra of the Pauli matrices that

$$\rho - U(t)\rho U(t)^\dagger = \sin^2(t)(\vec{r} - (\vec{r}\cdot\vec{n})\vec{n})\cdot\vec{\sigma} + \sin t\cos t(\vec{r}\times\vec{n})\cdot\vec{\sigma}. \tag{7}$$

We derive the conditions on $\vec{r}$ which guarantee the existence of t such that $p_{\mathrm{err,min}}(\rho, U(t)\rho U(t)^\dagger) = \delta$ is satisfied. Evaluating the trace norm of Eq.(7) allows one to write the following expression for $p_{\mathrm{err}}(\rho, U(t)\rho U(t)^\dagger)$:

$$\frac{1}{2} - \frac{1}{2}\|\sin^2 t(\vec{r} - (\vec{r}\cdot\vec{n})\vec{n}) + \sin t\cos t(\vec{r}\times\vec{n})\|. \tag{8}$$

The expression in (8) is equal to δ if and only if

$$\frac{1-2\delta}{\sqrt{\|\vec{r}\|^2-(\vec{r}\cdot\vec{n})^2}} = |\sin t|. \tag{9}$$

A value of t satisfying the above equation exists if and only if $\frac{1-2\delta}{\sqrt{\|\vec{r}\|^2-(\vec{r}\cdot\vec{n})^2}} \leq 1$. Using the Lemma, we have $\sqrt{\mathcal{F}(\rho,H)} = 2\|\vec{n}\times\vec{r}\| = 2\sqrt{\|\vec{r}\|^2-(\vec{r}\cdot\vec{n})^2}$, which was required. □

An immediate corollary of Theorem 1 is that pure qubit states are the only states for which there exists an orthogonalizing unitary evolution. In addition, it is clear from expanding the vector norm in Eq.(8) that the minimal error for distinguishing ρ and $\rho(t)$ occurs at $t=\pi/2$ (in units of ω_0^{-1}). It follows from Eq.(9) that if ρ saturates the inequality (6), then $p_{\mathrm{err}}(\rho, U(\pi/2)\rho U(\pi/2)^\dagger) = \delta$, i.e., $\rho(t)$ is $(1-\delta)$-distinguishable from ρ at time $t=\pi/2$.

Having derived the necessary and sufficient condition for a given qubit state to reach a $(1-\delta)$-distinguishable state under a given unitary evolution, we are equipped to find the set of states that evolve to $(1-\delta)$-distinguishable states in less time than the given state. In particular, it follows as a corollary of Theorem 1 that there are mixed states that evolve more quickly to $(1-\delta)$-distinguishable states than certain pure states, as long as $\delta > 0$. To make this clear, we take $H=\hbar\omega_0\sigma_z$ without loss of generality and first note that the maximal quantum Fisher information of all paths in the Bloch ball generated by H is achieved for the pure states $|0\rangle + e^{i\varphi}|1\rangle/\sqrt{2}$ ($\varphi\in[0,2\pi)$); these are the "fastest" time-evolving states, reaching $(1-\delta)$-distinguishable states in time $\sin^{-1}(1-2\delta)\leq\pi/2$. Now, consider a pure state $|\psi\rangle$ with Bloch vector given by angular parameters $(\theta_\psi,\varphi_\psi)$ on the Bloch sphere with $\theta_\psi = \sin^{-1}(1-2\delta)$ (Fig. 2). Then Eq.(9) implies that $\tau_\delta(|\psi\rangle, e^{-i\omega_0 t\sigma_z}, p_{\mathrm{err}}) = \pi/2$. It follows from Theorem 1 that the set S of states defined by $S=\{\rho|\sqrt{\mathcal{F}(\rho,H)}/2 \geq \sin\theta_\psi = 1-2\delta\}$ reach $(1-\delta)$-distinguishable states and do so in a time $t\in[\sin^1(1-2\delta),\pi/2]$, i.e., in a time less than or equal to the $(1-\delta)$-distinguishability time of $|\psi\rangle$. The set of states satisfying this condition lie in the spherical ring illustrated in Fig. 2. Hence, to find the set of states evolving faster than a given state ρ, one must only look for the states σ such that $\mathcal{F}(\sigma,H)\geq\mathcal{F}(\rho,H)$.

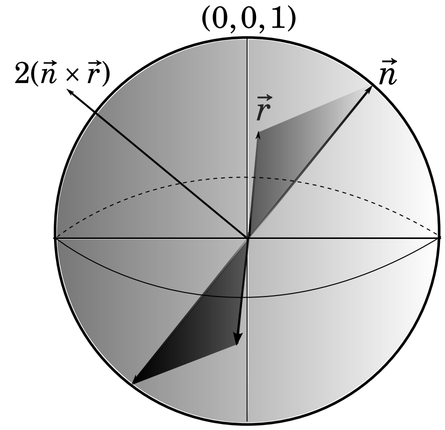

FIG. 1: The magnetic field vector $\vec{n}$, the Bloch vector $\vec{r}$, and the direction vector $2(\vec{n}\times\vec{r})$ of the symmetric logarithmic derivative plotted relative to the 2-sphere. The square root of the quantum Fisher information is equal to the operator norm $\|L\|$ of L which is twice the shaded area.

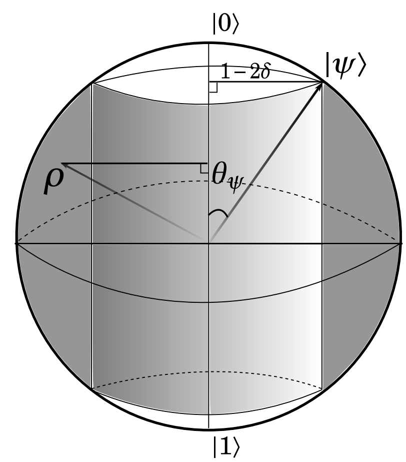

FIG. 2: Given a pure quantum state $|\psi\rangle$ with Bloch vector corresponding to polar angle θ_ψ and Hamiltonian $H=\hbar\omega_0\sigma_z$, the shaded region containing ρ represents those states with a shorter distinguishability time for all values of δ.

IV. DISTINGUISHABILITY ON MINIMAL TIME PATHS

As an example of utility of the notion of distinguishability time, we provide a simple solution of the problem of the quantum brachistochrone, i.e., given two states ρ_1, ρ_2 with equal Bloch vector magnitudes, to identify a Hamiltonian H_* generating a unitary path such that ρ_1 evolves to ρ_2 in the shortest possible time. Previous approaches to this problem make use of a variational principle on projective Hilbert space [20] or saturating an upper bound on the energy fluctuations in a 2-dimensional Hilbert subspace [21]. Our solution solves the brachistochrone for two mixed states with equal Bloch vector magnitudes and the pure state brachistochrone follows as a corollary when the Bloch vectors are on the Bloch sphere.

Theorem 2 Let ρ_1, ρ_2 be quantum states of a two-level system and have Bloch vectors $\vec{r_1}$, $\vec{r_2}$, respectively, with $\|\vec{r_1}\| = \|\vec{r_2}\|$. Then the Hamiltonian H_* having operator norm $\hbar\omega_0$ which takes ρ_1 to ρ_2 in the minimal time is given by

$$H_* = \hbar\omega_0 \frac{\vec{r_1}\times\vec{r_2}}{\|\vec{r_1}\times\vec{r_2}\|}\cdot\vec{\sigma}. \tag{10}$$

Proof The assumptions of the theorem come only from the observation that unitary dynamics of a mixed

state with Bloch vector magnitude R take place on a 2-dimensional sphere of radius R. To prove the Theorem, let $\omega_0 = 1$ and consider the ansatz $H_* = \hbar\vec{q}\cdot\vec{\sigma}$ with $\|\vec{q}\| = 1$. We assume that there exists t such that $\rho_2 = e^{-iH_*t/\hbar}\rho_1 e^{iH_*t/\hbar}$. Then the time evolved Bloch vector of ρ_1 satisfies

$$\cos(2t)\vec{r_1} - \sin(2t)(\vec{r_1}\times\vec{q}) + 2\sin^2(t)(\vec{r_1}\cdot\vec{q})\vec{q} = \vec{r_2}. \quad (11)$$

Taking the Euclidean dot product of Eq.(11) with $\vec{r_1}$ gives the equation

$$\sin^2 t = \frac{\|\vec{r_1}\|^2 - \vec{r_1}\cdot\vec{r_2}}{2\sin^2\theta_1\|\vec{r_1}\|^2} \quad (12)$$

where θ_1 is defined as the angle between $\vec{r_1}$ and $\vec{q}$. The time t is minimal when $\theta_1 = \pi/2$; we will call this time T. If θ_1 takes this value, then it is clear from Eq.(11) that $\vec{q}\cdot\vec{r_2} = 0$. Up to a factor of ± 1, this specifies $\vec{q} = \frac{\vec{r_1}\times\vec{r_2}}{\|\vec{r_1}\times\vec{r_2}\|}$. To finish the proof, we show that taking $H_* = \vec{q}\cdot\vec{\sigma}$ with $\vec{q} = \frac{\vec{r_1}\times\vec{r_2}}{\|\vec{r_1}\times\vec{r_2}\|}$ (i.e., taking the "+" sign) causes ρ_1 to unitarily evolve to ρ_2 in the minimal possible time allowed by Eq.(9). Since the minimal error probability for distinguishing ρ_1 and ρ_2 is $\delta = 1/2 - 1/4\|\vec{r_1} - \vec{r_2}\|$ and using $\|\vec{r_1}\| = \|\vec{r_2}\|$, we have $2(1-2\delta) = \|\vec{r_1} - \vec{r_2}\| = \sqrt{2(\|\vec{r_1}\|^2 - \vec{r_1}\cdot\vec{r_2})}$. For $\theta_1 = \pi/2$, $\mathcal{F}(\rho_1, H_*) = 4\|\vec{r_1}\|$. Hence, Eq.(12) shows that ρ_1 evolves to ρ_2 on the path generated by H_* in time

$$T = \sin^{-1}\frac{\|\vec{r}_1 - \vec{r}_2\|}{\sqrt{\mathcal{F}(\rho_1, H_*)}} = \sin^{-1}\frac{(1-2\delta)}{\sqrt{\mathcal{F}(\rho_1, H_*)}}. \quad (13)$$

Therefore, according to Eq.(9), H_* gives the shortest time unitary evolution taking ρ_1 to ρ_2. □

The form of Eq.(10) could have been anticipated by noting that the goal is to identify the unitary path connecting ρ_1 and ρ_2 that has greatest quantum Fisher information (i.e., greatest Bures line element [19]) and then noting that the Bures line element is just a Euclidean line element on Bloch vectors [22]. Hence, one seeks a great circle on the sphere of radius $\|\vec{r_1}\|$ connecting ρ_1 and ρ_2 in the Bloch ball. The quantum Fisher information on the path $e^{-itH_*/\hbar}\rho_1 e^{itH_*/\hbar}$ is

$$\mathcal{F}(\rho_1, H_*) = \frac{4(1 - \|r_1\|^2\cos^2(\varphi_{12}))}{\sin\varphi_{12}} \quad (14)$$

where $\varphi_{12}\in[0,\pi]$ is the angle between $\vec{r_1}$ and $\vec{r_2}$.

To obtain the pure state brachistochrone, consider two linearly independent pure states $|\psi_1\rangle$, $|\psi_2\rangle$ with $\langle\psi_1|\psi_2\rangle =: z \in \mathbb{R}$ to span a Hilbert subspace, form the orthonormal basis with elements

$$|e_\pm\rangle = |\psi_1\rangle \pm |\psi_2\rangle/\sqrt{2+2z}. \quad (15)$$

Then the state $|\psi_1\rangle\langle\psi_1|$ ($|\psi_2\rangle\langle\psi_2|$) has Bloch vector $\vec{n_1} = (\sqrt{1-z^2}, 0, z)$ ($\vec{n_1} = (-\sqrt{1-z^2}, 0, z)$) and, from Eq.(10), one has $H_* = -\hbar\omega_0\sigma_y$, where $\sigma_y = -i|e_+\rangle\langle e_-| +$ h.c.. Rewriting $-\sigma_y$ gives the quantum brachistochrone for two pure states:

$$H_* = \frac{-i\hbar\omega_0}{\sqrt{1-z^2}}\left(|\psi_1\rangle\langle\psi_2| - |\psi_2\rangle\langle\psi_1|\right). \quad (16)$$

Note also that under brachistochrone evolution to a pure state $|\psi_2\rangle$, an initial pure state $|\psi_1\rangle$ orthogonalizes to the state $\propto z|\psi_1\rangle - |\psi_2\rangle$ after a time $\pi/2$.

V. CAVITY QUBIT DISTINGUISHABILITY

In Theorem 1, we have established the necessary and sufficient condition for an arbitrary qubit state to reach a $(1-\delta)$-distinguishable state under unitary evolution in the Bloch ball. The restriction to unitary evolution is made here primarily on account of the relevance of such maps for quantum computations, where Stinespring's dilation theorem [23] allows one to consider any completely positive, trace-preserving map as a unitary map on an amplified system consisting of the original system and some ancilla qubits. However, within current physical realizations of quantum computers, it is not a trivial task to simply append the needed ancilla qubits and carry out the unitary map; it may therefore often be more efficient to focus efforts on engineering dissipative quantum evolutions that optimally take initial states to the desired final states in the Bloch ball.

With this in mind, we now consider non-unitary distinguishability dynamics by introducing a resonant interaction of the qubit with a single-mode electromagnetic field, e.g., in a cavity quantum electrodynamics (CQED) experiment. We shall analyze the distinguishability of the initial qubit state from its reduced time evolved counterpart. In the case of unitary evolution, the only way to decrease the distinguishability time for a given initial qubit state is to increase the energy scale $\hbar\omega_0$ of the qubit. Intuitively, one expects that a more energetic quantum state moves faster in Hilbert space and this is verified by the inverse scaling of the ML$_\perp$ and MT$_\perp$ bounds with the operator norm of the Hamiltonian. However, for a qubit resonantly coupled to a lossless single-mode electromagnetic field, coherent control of the initial field state can allow the qubit distinguishability time to be tuned for *any* given value of the qubit energy splitting.

We calculate the quantity $p_{\mathrm{err}}(\rho_S(t), \rho_S(0))$ where $\rho_S(0)$ is the initial qubit state, $|\psi\rangle$ is the initial (pure) state of the field, and $\rho_S(t) = \mathrm{tr}_E(V(t)(\rho_S(0)\otimes|\psi\rangle\langle\psi|)V(t)^\dagger)$ is the reduced time-evolved state of the qubit (Fig.3). The unitary $V(t)$ is the time-evolution operator generated by the Jaynes-Cummings Hamiltonian [24]

$$H_{JC} := \hbar\omega_0(a^\dagger a + \frac{\sigma_z}{2}) + g(a^\dagger\sigma_- + a\sigma_+) \quad (17)$$

at field frequency $\omega_0 = 1$, zero detuning, and qubit-field coupling $g = \omega_0/20$. We parametrize the initial qubit

a)

b)

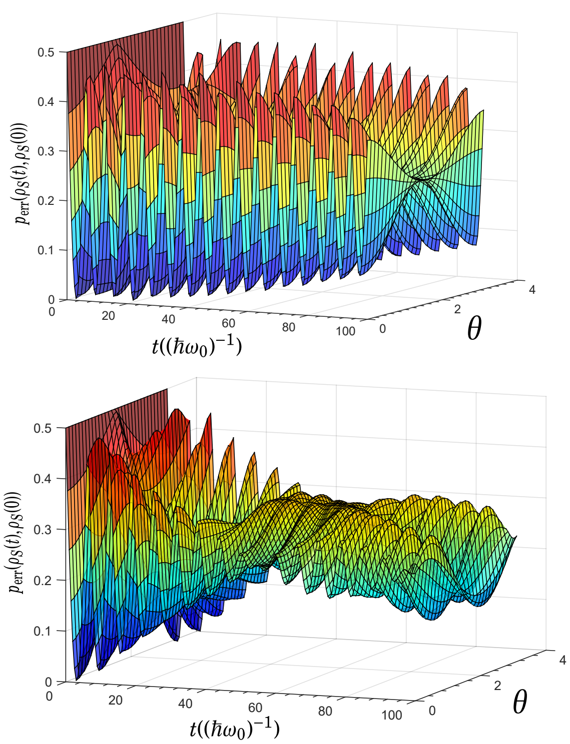

FIG. 3: Minimal error probability for distinguishing an initial pure state in the xz-plane of the Bloch sphere from $\mathrm{tr}_E(U(t)(\rho_S \otimes |\psi\rangle\langle\psi|)U^\dagger(t))$ for $t \in [0, 100\omega_0^{-1}]$ with $|\psi\rangle = |\alpha\rangle$ and a) $\alpha = 3$ b) $\alpha = 1$. c) Reduced distinguishability dynamics of qubit initially in $\rho_S(0) = |0\rangle\langle 0|$ coupled to various initial field states: $|\psi\rangle = |\alpha\rangle$ (black, dashed); $|\psi\rangle = |e_0\rangle\langle e_0|$ given by Eq.(20) (black, solid); $|\psi\rangle \propto |\alpha\rangle + |-\alpha\rangle$ (blue); $|\psi\rangle \propto |\alpha\rangle - |-\alpha\rangle$ (red). $\alpha = 3$ for all curves.

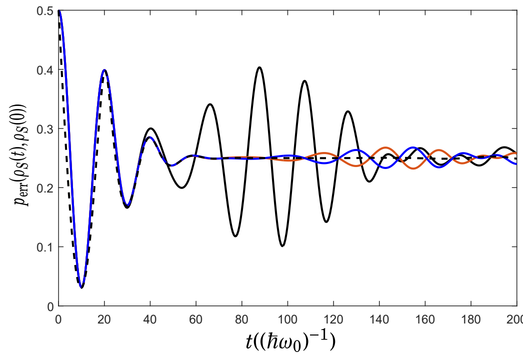

FIG. 4: Reduced distinguishability dynamics of qubit initially in $\rho_S(0) = |0\rangle\langle 0|$ coupled to various initial field states: (black, dashed) $|\psi\rangle = |\alpha\rangle$; (black, solid) $|\psi\rangle = |e_0\rangle\langle e_0|$ given by Eq.(20); (blue) $|\psi\rangle \propto |\alpha\rangle + |-\alpha\rangle$; (red) $|\psi\rangle \propto |\alpha\rangle - |-\alpha\rangle$ ($\alpha = 3$ for all curves).

state by

$$\rho_S(0) = \frac{\mathbb{I}}{2} + \frac{\|\vec{r}\|}{2}(\cos(\theta)\sigma_x + \sin(\theta)\sigma_z) \tag{18}$$

for ease of visualization. The reduced time-evolved qubit state $\rho_S(t)$ is calculated with the Kraus representation of the reduced dynamics defined in the Fock state representation,

$$\rho_S(t) = \sum_{n=0}^{100} E_n(t)\rho_S(0)E_n(t)^\dagger, \tag{19}$$

where $E_n(t) := \langle n|V(t)|\psi\rangle$ is a bounded operator on the qubit Hilbert space. We employ $N = 100$ Kraus operators, corresponding to contributions from field photon Fock states up to $N = 100$ and vary the initial product state of qubit and field as described below.

Figs. 3a) and 3b) show the time dependence of the minimal error probability for distinguishing the time-evolved reduced qubit state from the initial qubit state when the initial field state is a coherent state with variable intensity, i.e., $|\psi\rangle = |\alpha\rangle$, $\alpha \in \mathbb{C}$. Two effects of the field intensity $|\alpha|^2$ on the minimal error of optimal discrimination between $\rho_S(0)$ and $\rho_S(t)$ are evident here: i) increasing the field energy causes large amplitude oscillations in the distinguishability with respect to time, and ii) the timescale of revivals of distinguishability is extended by increasing the field energy (Fig. 3a)). In the frame corotating with the qubit, the timescale of the fast oscillations of $p_{\rm err}$ is governed by the quantum Rabi frequency $f_n := g\sqrt{n}/\hbar$ associated to each Fock component of the initial field state $|\psi\rangle$. In the limit of zero detuning, the relevant timescale of revivals of the distinguishability for an initially coherent field is $\mathcal{O}(\hbar|\alpha|/g)$.

The reduced distinguishability dynamics of a two-level quantum system in an arbitrary environment are simplified by the fact that a two-level system can only undergo two dynamical processes: 1) emission/absorption (i.e., losing or gaining energy), and 2) dephasing (i.e., gaining or losing quantum coherence in a chosen basis). The form of the qubit-field interaction term in the Jaynes-Cummings model, Eq.(17), suggests that the timescales associated to oscillations, collapses, and revivals of distinguishability should depend on specific features of the photon number distribution of the initial field state $|\psi\rangle$. For example, starting with an initially excited qubit state $\rho_S(0) = |0\rangle\langle 0|$, the off-diagonal terms of $\rho_S(0) - \rho_S(t)$ depend linearly on the product $\langle n \pm 1|\psi\rangle\overline{\langle n|\psi\rangle}$ and its complex conjugate. By preparing the field in an even or odd coherent state $|\psi\rangle \propto |\alpha\rangle \pm |-\alpha\rangle$, these dephasing contributions to the distinguishability vanish. This is illustrated in Fig. 4, where the solid red and blue traces reveal that at a constant intensity $|\alpha|^2 = 9$, the qubit exhibits distinguishability revivals of greater amplitude and at shorter times in the even/odd coherent fields than in the coherent field.

The damping (diagonal) contribution to $\rho_S(0) - \rho_S(t)$ can also be tuned in order to change the timescale of distinguishability revivals. Consider the initial field state

$|e_0\rangle$ given by

$$|e_0\rangle := \frac{|\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle}{2e^{-|\alpha|^2/2}\sqrt{2\cosh|\alpha|^2 + 2\cos|\alpha|^2}} \tag{20}$$

which exhibits a photon number distribution having support on n such that $n = 0 \bmod 4$ [25]. Like the even and odd coherent states, this field state does not provide a dephasing contribution to $\rho_S(0) - \rho_S(t)$. However, for this state, only the Kraus operations $E_m(t)$ with $m \in \{1, 3\} \bmod 4$ contribute to the dynamics. In the solid black trace of Fig. 3c), this feature is seen to prevent the complete destructive interference of quantum Rabi frequencies from occurring at short timescales.

Note that in this work, we do not consider the reduced distinguishability dynamics of the field, but focus solely on the qubit dynamics. Analysis of the photon statistics, field entropy, and other physical characteristics of the reduced state of the field were analyzed in Ref.[26].

An analog of Theorem 1 for the reduced qubit distinguishability dynamics would require an analytical relation between the time-dependent quantity $p_{\text{err}}(\rho_S(0), \rho_S(t))$ and the time-dependent quantum Fisher information of the qubit. In Fig. 5, we demonstrate instead by example that given a pure state with Bloch vector not orthogonal to the z-axis, there are mixed states that evolve to a $(1-\delta)$-distinguishable state faster than the given pure state even in the presence of nonunitary dynamics. As expected from the linear field-qubit coupling in the Hamiltonian of Eq.(17), the quantum state of the field plays an important role. For δ in the range ~ 0.15 to 0.50, the mixed states on the x-axis with Bloch vector magnitude $9/10$ and initial field states $|\alpha\rangle$ and $|e_0\rangle$ (red and black dashed lines, respectively) are seen to reach their $(1-\delta)$-distinguishable states *faster* than the pure state at polar angle $3\pi/8$ in the xz-plane with initial field state $|\alpha\rangle$ (black solid line). For smaller, but still intermediate, vales of δ the pure state evolves faster than the mixed state with initial field $|e_0\rangle$ but slower than the mixed state with initial field $|\alpha\rangle$. For small δ, the pure state is faster than both mixed states.

This result is relevant to realistic quantum computations because it shows that given a known fidelity loss $\delta > 0$, i.e., given knowledge of the error rate of the unitary gates, of the environment of the computer, etc., one is able to operate the computer optimally using mixed states. For example, imperfectly prepared qubit states could thereby be used to increase the processing speed of a lossy quantum computer. This strategy not only speeds up the quantum computer, but also reduces the change in entropy associated with any quantum computation.

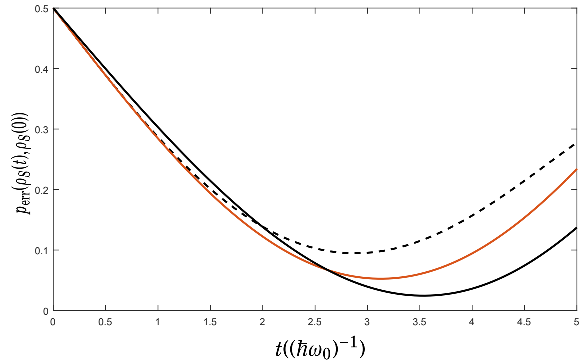

FIG. 5: Minimal error probability for distinguishing various initial states $\rho_S(0)$ of a qubit in the xz-plane of the Bloch ball from their respective reduced, time-evolved states $\rho_S(t)$ given initial field state $|\psi\rangle$. (Black solid) $\rho_S(0)$ having polar angle $3\pi/8$ and $\|\vec{r}\| = 1$, $|\psi\rangle = |\alpha\rangle$; (red) $\rho_S(0)$ having polar angle $\pi/2$ and $\|\vec{r}\| = 9/10$, $|\psi\rangle = |\alpha\rangle$; (black dashed) $\rho_S(0)$ having polar angle $\pi/2$ and $\|\vec{r}\| = 9/10$, $|\psi\rangle = |e_0\rangle$. $\alpha = 3$ in all cases.

VI. CONCLUSION

We have introduced the formal notion of distinguishability time as a generalization of the orthogonalization time appearing in rigorous definitions of the quantum speed limit and solved for the distinguishability time of unitary evolution of a two-level system (Eq.(9)) in the case that quantum binary distinguishability is used as the discrimination procedure. In particular, by bounding the quantum Fisher information from below, we have derived a necessary and sufficient condition on a quantum state ρ and Hamiltonian H such that ρ will evolve in time $t \geq 0$ to a state $e^{-iHt}\rho e^{iHt}$ from which it is distinguishable with maximal success probability $1-\delta$ for any $\delta \in [0, 1/2]$. As a corollary of this condition, we determined the set of quantum states that evolve to $(1-\delta)$-distinguishable states faster than a given quantum state under unitary time-evolution. The formalism developed in Section III was used in a simple proof of the quantum brachistochrone for a two-level system.

We have extended the distinguishability analysis of time-evolved states to the reduced Liouville-von Neumann dynamics of a two-level system resonantly coupled, via the Jaynes-Cummings interaction, to a single-mode electromagnetic field. The timescale of revivals of qubit state distinguishability under non-unitary dynamics induced by the quantum state of the field was found to depend on both the strength of the field and the photon number statistics. In the context of CQED-based quantum operations, these results suggest the possibility of engineering the field state and the field energy to tune the distinguishability dynamics. Calculation of the reduced distinguishability dynamics of a qubit under more general qubit-field interactions will be made in future work.

Acknowledgments

This work was supported by National Science Foundation Grant No. CHE-1213141.

[1] S. Lloyd, Nature **406**, 1047 (2000).
[2] A. Kitaev, A. Shen, and M. Vyalyi, *Classical and Quantum Computation* (American Mathematical Society, 1999).
[3] J. Bergou, U. Herzog, and M. Hillery, Lect. Notes Phys. **649**, 417 (2004).
[4] B. Samsonov, Phys. Rev. A **80**, 052305 (2009).
[5] A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[6] T. Volkoff and K. Whaley, Phys. Rev. A **90**, 062122 (2014).
[7] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. A **67**, 052109 (2003).
[8] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press , New York, 1976).
[9] S. Morley-Short, L. Rosenfeld, and P. Kok, Phys. Rev. A **90**, 062116 (2014).
[10] V. Dodonov and A. Dodonov, arXiv p. 1504.00862v1 (2015).
[11] S. Deffner and E. Lutz, Phys. Rev. Lett. **111**, 010402 (2013).
[12] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Phys. Rev. Lett. **110**, 050402 (2013).
[13] L. Mandelstam and I. Tamm, J. Phys. USSR **9**, 249 (1945).
[14] J. Anandan and Y. Aharonov, Phys. Rev. Lett. **65**, 1697 (1990).
[15] N. Margolus and L. Levitin, Physica D **120**, 188 (1996).
[16] N. Horesh and A. Mann, J. Phys. A.: Math. Gen. **31**, L609 (1998).
[17] Z. Jiang, Phys. Rev. A **89**, 032128 (2014).
[18] J. Liu, X.-X. Jing, and X. Wang, Sci. Rep. **5**, 8565 (2015).
[19] S. Braunstein and C. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[20] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Phys. Rev. Lett. **96**, 060503 (2006).
[21] A. Mostafazadeh, Phys. Rev. A **79**, 014101 (2009).
[22] M. Hubner, Phys. Lett. A **163**, 239 (1992).
[23] W. Stinespring, Proc. Am. Math. Soc. **6**, 211 (1955).
[24] A. Klimov and S. Chumakov, *Group theoretical methods in quantum optics* (Wiley-VCH, 2009).
[25] T. Volkoff, arXiv p. 1504.07123 (2015).
[26] V. Bužek, H. Moya-Cessa, P. Knight, and S. Phoenix, Phys. Rev. A **45**, 8190 (1992).